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Torsion and Related Concepts: An Introductory Overview

Borut Gogala

505-2200 Roche Court Mississauga, Ontario

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The aim of this paper is to provide an overview of all the basic aspects of the torsion of a manifold, with particular stress on the expressions in an anholonomic basis. After a brief review of anholonomic bases and Koszul covariant derivative, we show how the expressions for the torsion and the Riemann tensors in a general (anholonomic) basis arise from their expressions in a coordinate basis. We further derive the expression for the contortion tensor, which arises from the requirement that an affine connection with torsion be metric (preserving). The latter requirement is related to the equivalence principle, whose mathematical aspects in a manifold with torsion are discussed next. Finally, we derive the expression for the distortion tensor, which is an analog of the curvature tensor but arising from the torsion rather than the metric tensor.

1. INTRODUCTION

The concept of torsion of a differentiable manifold is not well known or, at best, not well understood by most students of relativity. There seem to be two reasons for this situation. The first is historical: neither differential geometry in its early development (which grew out of studying the hypersurfaces embedded in an Euclidean space), nor general relativity theory had any use for torsion. Although discovered by Cartan in the 1920s (Cartan, 1922, 1923, 1924, 1925), torsion has never been studied much by relativists until rather recently (see the review article by Hehl et al., 1976).

The second reason is that torsion, although conceptually rather simple, has never been explained in an organized, lucid, and complete fashion. The most that elementary textbooks tell about torsion is that it is the antisymmetric part of an affine connection, whereas advanced monographs on differential geometry introduce torsion axiomatically, without giving the reader any help to get the feeling for it. Schouten's (1954) book,

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on the other hand, provides too much information, so that it is difficult to extract the essentials and gain an overview; this is so particularly because the book does not use the modern (Koszul) approach to differential geometry.

The purpose of this paper is to correct this omission, and explain the concept of torsion and related concepts in the style of Misner (1969), and thus hopefully to serve as a supplement to that work as well as to the book by Misner, Thorne, and Wheeler (1973). Both of these works which unfortunately almost completely ignore the concept of torsion.

First, we recall the difference between a coordinate and an anholonomic basis, which is much more important for studying a manifold with torsion than when a manifold is endowed with metric only.

A *coordinate basis* is a set of n linearly independent vectors, defined in each point of the manifold, which are tangent to the n coordinate lines, which pass through that point and belong to a coordinate system (also called the global coordinate system or natural coordinate system) imposed on the manifold. The transformation coefficients between two coordinate bases, belonging to two different coordinate systems

$$\mathbf{e}_{K'}=h_{K'}{}^{L}\mathbf{e}_{L}, \qquad \mathbf{e}_{L}=h^{K'}{}_{L}\mathbf{e}_{K'} \tag{1.1}$$

are partial derivatives:

$$h_{K'}{}^{L}=\frac{\partial x^{L}}{\partial x^{K'}} \qquad h^{K'}{}_{L}=\frac{\partial x^{K'}}{\partial x^{L}} \tag{1.2}$$

Such transformations are thus integrable or *holonomic*, and consequently other coordinate bases, obtained from the application of such transformations on a coordinate basis, are also coordinate bases.

If the transformation coefficients are not partial derivatives, then we get from a coordinate basis $\mathbf{e}_L$ another basis $\mathbf{e}_k$ (we will use upper case indices to denote a coordinate basis, and lower case indices to denote an anholonomic basis, which includes a coordinate basis as a special case), which is called *anholonomic* or *nonintegrable* because it has been obtained from a coordinate basis through a nonintegrable or anholonomic transformation:

$$\mathbf{e}_{k}=h_{k}{}^{L}\mathbf{e}_{L}, \qquad \mathbf{e}_{L}=h^{k}{}_{L}\mathbf{e}_{k} \tag{1.3}$$

This basis is also called a noncoordinate basis or, in the French literature, *repère mobile*. Although it is always possible to find a coordinate basis which will coincide with an anholonomic basis in one point, that is, locally, there is no coordinate system which would correspond to the anholonomic basis globally. Nevertheless, we can, in an anholonomic basis, still define objects which correspond to partial derivatives in a coordinate basis by

$$(\mathbf{e}_k f) = f_{/k} = \partial_k f = h_k{}^L \frac{\partial f}{\partial x^L} = h_k{}^L(\mathbf{e}_L f) \tag{1.4}$$

These objects are sometimes referred to as *Pfaffian derivatives* (Zorawski, 1967, p. 2).

In general the basis vectors of an anholonomic basis do not commute. That is, their commutators (Lie brackets) (see Misner et al., 1973, p. 204, 206, 235; Misner, 1969, p. 123) are not zero, but

$$[\mathbf{e}_k, \mathbf{e}_l] = c^m{}_{kl}\mathbf{e}_m \tag{1.5}$$

The quantities $c^m{}_{kl}$ are called *commutation coefficients* (Misner et al., 1973, p. 204, 206, 239); the name components of the *object of anholonomity* (Schouten, 1954, p. 100; Zorawski, 1967, p. 2; Golab, 1974, p. 140) is also used. We recall that a commutator of two vectors can be expressed in terms of their components (see Misner et al., 1973, p. 206, 236; Misner, 1969, p. 123) as

$$[\mathbf{u}, \mathbf{v}] = (u^m v^n{}_{/m} - v^m u^n{}_{/m})\mathbf{e}_n \tag{1.6}$$

Next we give a brief overview of the notion of covariant derivative in its modern, basis-independent, so-called *Koszul* form (Spivak, 1970, p. 6-1), which was introduced by Nomizu (1954, p. 35) on Koszul's advice. A covariant derivative or an affine connection is an operator ∇ that assigns to each pair of vector fields $\mathbf{X}$ and $\mathbf{Y}$, a third vector field $\nabla_{\mathbf{X}}\mathbf{Y}$ that satisfies the following axioms (Misner, 1969, p. 128; Misner et al., 1973, p. 252):

$$\nabla_{f\mathbf{X}_1 + g\mathbf{X}_2}\mathbf{Y} = f\nabla_{\mathbf{X}_1}\mathbf{Y} + g\nabla_{\mathbf{X}_2}\mathbf{Y} \qquad \text{(linearity)} \tag{1.7a}$$

$$\nabla_{\mathbf{X}}(\mathbf{Y}_1 + \mathbf{Y}_2) = \nabla_{\mathbf{X}}\mathbf{Y}_1 + \nabla_{\mathbf{X}}\mathbf{Y}_2 \qquad \text{(additivity)} \tag{1.7b}$$

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + (\partial_{\mathbf{X}} f)\mathbf{Y} \qquad \text{(chain rule for differentiation)} \tag{1.7c}$$

We will write briefly

$$\nabla_k = \nabla_{\mathbf{e}_k} \tag{1.8}$$

for the covariant derivative along the basis vector $\mathbf{e}_k$. In a given basis, we can express the covariant derivative of a basis vector in terms of its components as

$$\nabla_k \mathbf{e}_i = \Gamma^j{}_{ik}\mathbf{e}_j \tag{1.9}$$

We must make a remark here about the notation used for the lower two indices of the components of an affine connection. We use the first index to denote the differentiated vector, and the second one to denote the differentiating vector. In the literature, the opposite choice seems to be more frequent (see, e.g., Hehl et al., 1976). We, however, adopted the above choice, which has also been used by Misner (1969), Misner et al. (1973), Einstein (1955, p. 143), Schroedinger (1950, p. 40), and others, for the following psychological reason. In component form, the covariant derivative of an arbitrary vector $\mathbf{v}$ is written as

$$\nabla_m \mathbf{v} = v^k{}_{;m}\mathbf{e}_k = \left(v^k{}_{/m} + v^l \Gamma^k{}_{lm}\right)\mathbf{e}_k \tag{1.10}$$

With our notation, the index of the differentiating vector is the last one in both terms in the brackets.

Of course, when torsionless (symmetric) connections are discussed, as is the case in most treatments on differential geometry and general relativity theory, the choice of notation does not matter as much as in our treatment, dealing explicitly with connections with torsion.

2. THE CONCEPT OF TORSION

We now turn our attention to the concept of torsion. The most that elementary textbooks on differential geometry say about torsion is that it is the antisymmetric part of an affine connection. It is easy to show that such a quantity behaves as a tensor under the holonomic (integrable) transformations of the coordinate basis, and one thus speaks about the torsion tensor.

When one wants to extend the concept of torsion to an anholonomic basis, the above definition is no longer satisfactory. First, the symmetry properties of the affine connection components are not preserved under an anholonomic transformation of the basis. As an example we consider the components of the so-called *Levi-Civita* (Bishop and Goldberg, 1968, p. 241; Spivak, 1970, pp. 6-17; Hayashi, 1976) *connection*, that is, the usual metric (preserving) torsionless connection of a Riemannian manifold, whose general form is [Misner et al., 1973, equation (13.23)]

$$\hat{\Gamma}_{k\,lm} = \tfrac{1}{2}(g_{kl/m} + g_{km/l} - g_{lm/k}) + \tfrac{1}{2}(c_{m\,kl} + c_{l\,km} - c_{k\,lm}) = [k, lm] + \gamma_{m\,kl} \tag{2.1}$$

The first term is usually called the *Christoffel symbol*, and the second the *Ricci rotation coefficient* (Levi-Civita, 1929, pp. 268–272 or Section X.3; Weatherburn, 1950, pp. 98–102 or Sections 54–57; Schouten 1954, p. 171; Gerretsen, 1962, pp. 119, 120 or Section 7.7.3; Golab, 1974; pp. 255–258 or

Section 89). We observe that this connection is symmetric in the last two indices only in a coordinate basis, but loses its symmetry in an anholonomic basis. In the extreme case of an orthonormal anholonomic basis, it becomes antisymmetric in the first two indices (note that in an orthonormal basis, the partial derivatives of the metric tensor components are identically zero).

Secondly, the antisymmetric part of an affine connection no longer transforms as a tensor, as we will now show.

In a coordinate basis, we have

$$\mathbf{T}(\mathbf{e}_K, \mathbf{e}_L) = T^M{}_{KL}\mathbf{e}_M = (\Gamma^M{}_{LK} - \Gamma^M{}_{KL})\mathbf{e}_M = \nabla_K \mathbf{e}_L - \nabla_L \mathbf{e}_K \tag{2.2}$$

We now transform this expression into an anholonomic basis (lower case indices)

$$T^m{}_{kl} h^k{}_K h^l{}_L \mathbf{e}_m = h^k{}_K \nabla_k (h^l{}_L \mathbf{e}_l) - h^l{}_L \nabla_l (h^k{}_K \mathbf{e}_k) \tag{2.3}$$

where the axiom (1.7a) was applied to the right-hand side. We then apply axioms (1.7b) and (1.7c) to get

$$T^m{}_{kl} h^k{}_K h^l{}_L \mathbf{e}_m = h^k{}_K h^l{}_L (\nabla_k \mathbf{e}_l - \nabla_l \mathbf{e}_k) + h^k{}_K h^l{}_{L/k} \mathbf{e}_l - h^l{}_L h^k{}_{K/l} \mathbf{e}_k \tag{2.4}$$

We also express the commutator of the coordinate basis vectors $[\mathbf{e}_K, \mathbf{e}_L]$, which is of course zero, in the anholonomic basis using (1.4), (1.6), and (1.7):

$$\begin{aligned} 0 = [\mathbf{e}_K, \mathbf{e}_L] &= h^k{}_K (h^l{}_L \mathbf{e}_l)_{/k} - h^l{}_L (h^k{}_K \mathbf{e}_k)_{/l} \\ &= h^k{}_K h^l{}_{L/k} \mathbf{e}_l - h^l{}_L h^k{}_{K/l} \mathbf{e}_k + h^k{}_K h^l{}_L [\mathbf{e}_k, \mathbf{e}_l] \end{aligned} \tag{2.5}$$

By inserting (2.5) into (2.4), we then get

$$h^k{}_K h^l{}_L T^m{}_{kl} \mathbf{e}_m = h^k{}_K h^l{}_L (\nabla_k \mathbf{e}_l - \nabla_l \mathbf{e}_k - [\mathbf{e}_k, \mathbf{e}_l]) \tag{2.6}$$

If we now define torsion generally as

$$\mathbf{T}(\mathbf{e}_k, \mathbf{e}_l) = T^m{}_{kl} \mathbf{e}_m = (\nabla_k \mathbf{e}_l - \nabla_l \mathbf{e}_k - [\mathbf{e}_k, \mathbf{e}_l]) = (\Gamma^m{}_{lk} - \Gamma^m{}_{kl} - c^m{}_{kl})\mathbf{e}_m \tag{2.7}$$

instead of (2.2), then the components $T^m{}_{kl}$ will always transform as components of a third-rank tensor, no matter what the basis or the transformation is. This is evidently due to the proper combination of the covariant derivative and the commutator in such a way that all the terms that would spoil the tensor transformation properties cancel out.

We can use a similar approach to derive the expression (2.12) for the Riemann tensor. For the sake of completeness, we also give that derivation. In a coordinate basis, we have

$$\mathbf{R}(\mathbf{e}_K, \mathbf{e}_L)\mathbf{e}_N = R^M{}_{N\,KL}\mathbf{e}_M = \nabla_K \nabla_L \mathbf{e}_N - \nabla_L \nabla_K \mathbf{e}_N \tag{2.8}$$

which after the transformation to an anholonomic basis becomes

$$\begin{aligned} R^m{}_{n\,kl} h^n{}_N h^k{}_K h^l{}_L \mathbf{e}_m & \\ &= h^k{}_K\left(h^l{}_L \nabla_k \nabla_l \mathbf{e}_N + h^l{}_{L/k} \nabla_l \mathbf{e}_N\right) - h^l{}_L\left(h^k{}_K \nabla_l \nabla_k \mathbf{e}_N + h^k{}_{K/l} \nabla_k \mathbf{e}_N\right) \\ &= h^k{}_K h^l{}_L(\nabla_k \nabla_l - \nabla_l \nabla_k - \nabla_{[\mathbf{e}_k, \mathbf{e}_l]}) h^n{}_N \mathbf{e}_n \end{aligned} \tag{2.9}$$

The last equality is a consequence of the fact that

$$\nabla_{[\mathbf{e}_K, \mathbf{e}_L]} = h^k{}_K h^l{}_{L/k} \nabla_l - h^l{}_L h^k{}_{K/l} \nabla_k + h^k{}_K h^l{}_L \nabla_{[\mathbf{e}_k, \mathbf{e}_l]} \tag{2.10}$$

also equals zero, which follows from (2.5) and (1.7a). An explicit calculation shows that $h^n{}_N$ will be unaffected, when pulled through the combination of the covariant differentiation operators in (2.9), so that (2.9) then becomes

$$R^m{}_{n\,kl} h^n{}_N h^k{}_K h^l{}_L \mathbf{e}_m = h^k{}_K h^l{}_L h^n{}_N (\nabla_k \nabla_l - \nabla_l \nabla_k - \nabla_{[\mathbf{e}_k, \mathbf{e}_l]}) \mathbf{e}_n \tag{2.11}$$

Thus we have the definition of Riemann

$$\mathbf{R}(\mathbf{e}_k, \mathbf{e}_l)\mathbf{e}_n = R^m{}_{n\,kl}\mathbf{e}_m = (\nabla_k \nabla_l - \nabla_l \nabla_k - \nabla_{[\mathbf{e}_k, \mathbf{e}_l]})\mathbf{e}_n \tag{2.12}$$

which is valid in any basis.

The above two derivations can be found, in a more compact form, in Nomizu (1954, pp. 37, 38) and in Choquet-Bruhat (1968, pp. 237–240). Unfortunately, most of the modern books on differential geometry only introduce (2.7) and (2.12) axiomatically, leaving the reader without a feeling for the quantities involved, and wondering how they ever came about; most of them do not even bother to explain that the reason for both definitions is their tensor transformation properties.

By comparing (1.5) and (2.7), one can immediately see the difference between the torsion and the commutator. Their components may indeed be equal (up to a sign) in a specially chosen, anholonomic basis (see below), when all the components of the affine connection are zero, but such an equality is just a coincidence. In a holonomic basis, the components of the commutator will always vanish; they depend on the choice of the basis. On the other hand, the components of torsion will, in general,

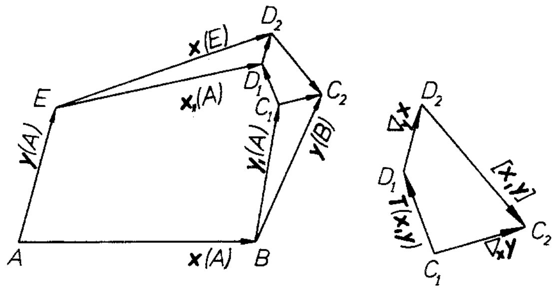

Fig. 1

not vanish in any basis, unless the connection is torsionless. Torsion is thus an intrinsic property of a manifold, independent of the choice of the basis.

The difference between both quantities is also obvious from the pictorial representation[1] shown in Figure 1.

The torsion vector measures the difference between two parallel transported basis vectors [$\mathbf{x}_{\|}(A)$ and $\mathbf{y}_{\|}(A)$]; it tells how much an infinitesimal parallelogram fails to close.[2]

The commutator, however, is "the closer of the curve" (Misner et al., 1973, p. 236), curves being the segments of coordinate lines; it measures the difference between two basis vectors [$\mathbf{x}(E)$ and $\mathbf{y}(B)$] of the coordinate quadrilaterals.

3. THE CONTORTION TENSOR

The requirement that the metric tensor be covariantly constant or stationary ($\nabla \mathbf{g}=0$) imposes an additional restriction on an affine connection with torsion, or rather on its components. A connection satisfying this requirement is called *metric* or *metric preserving;* in modern books on differential geometry the term "connection compatible with metric" is also used (Bishop and Goldberg, 1968, p. 238; Spivak, 1970, pp. 6-14, -15, -16). As will be shown below, this requirement follows from the equivalence

[1] On the basis of a similar picture Misner et al. (1973, p. 250) argued that the torsion of the space-time manifold must vanish because of the equivalence principle. Their argument is wrong, however, because the picture on the top of p. 250, which is critical for the incorrect conclusion, is constructed from the picture at the bottom of p. 249. This picture is true only if the torsion of the manifold is zero. We thus have a *circulus vitiosus*. In addition, they talk about geodesics without first defining the metric tensor and its being covariantly constant. The same observation has been made by J. M. Nester, University of Maryland (Hehl, 1977).

[2] This may be the source of the word "torsion." A firm parallelogram (e.g., a wire frame) will break, if the surrounding material (e.g., plasticine) is subjected to a torsional deformation along an axis either parallel or perpendicular to the plane of the parallelogram.

principle. In order to find the expression for the components of this connection, we could use the same method as the one used to derive equation (2.1) for the Levi-Civita connection (see Choquet-Bruhat, 1968, p. 242; also Misner et al., 1973, exercise 13.4). For a connection with torsion, such a derivation is given (in a coordinate basis only) for example in Schroedinger (1950, pp. 65, 66), although Misner (1969, p. 135) hints at it too.

We use a more elegant approach instead. The difference between *Cartan's*[3] *connection* (metric-preserving connection with torsion) and the Levi-Civita connection is a tensor

$$S_k{}^m{}_l \mathbf{e}_m = (\Gamma^m{}_{lk} - \hat{\Gamma}^m{}_{lk})\mathbf{e}_m = \nabla_k \mathbf{e}_l - \hat{\nabla}_k \mathbf{e}_l \tag{3.1}$$

Since the Levi-Civita connection is torsionless, we can write (2.7) as

$$T^m{}_{kl}\,\mathbf{e}_m = (\nabla_k - \hat{\nabla}_k)\mathbf{e}_l - (\nabla_l - \hat{\nabla}_l)\mathbf{e}_k = (S_k{}^m{}_l - S_l{}^m{}_k)\mathbf{e}_m \tag{3.2a}$$

or

$$T_{m\,kl} = S_{k\,ml} - S_{l\,mk} \tag{3.2b}$$

So far, the tensor $S_{m\,kl}$ is completely arbitrary.

We consider now the expression

$$\nabla_m(\mathbf{e}_k \cdot \mathbf{e}_l) - \hat{\nabla}_m(\mathbf{e}_k \cdot \mathbf{e}_l) \tag{3.3}$$

On the one hand, it is identically zero because the covariant derivative of a scalar is equal to its partial derivative, no matter what the connection is. On the other hand, the expression (3.3) is equal to

$$\nabla_m(\mathbf{e}_k \cdot \mathbf{e}_l) - \hat{\nabla}_m(\mathbf{e}_k \cdot \mathbf{e}_l) = \mathbf{e}_k \cdot (\nabla_m - \hat{\nabla}_m)\mathbf{e}_l + \mathbf{e}_l \cdot (\nabla_m - \hat{\nabla}_m)\mathbf{e}_k = S_{m\,kl} + S_{m\,lk} \tag{3.4}$$

because the covariant derivative of the dot ($\cdot$), that is, of the metric, is zero (cf. Misner, 1969, p. 135; also Misner et al., 1973, p. 314). Thus the tensor $S_{m\,kl}$ is antisymmetric in its last two indices, and so has the same number of components as the torsion tensor, that is, $n^2(n-1)/2$, and can be expressed in terms of the torsion tensor as

$$S_{m\,kl} = \tfrac{1}{2}(T_{m\,kl} + T_{l\,km} - T_{k\,lm}) \tag{3.5}$$

This tensor was named the *contortion* tensor by Hehl and Datta (1971).

[3]Such use of the term "Cartan connection" is frequent in physics papers dealing with torsion. The term is used also to mean a matrix of differential forms, giving the information about the parallel transport in Cartan's formulation of differential geometry in the language of differential forms (Spivak, 1970, pp. 7-32).

We see that the metric (preserving) connection is uniquely determined by the metric and the torsion fields on the manifold

$$\Gamma_{k\ lm} = \hat{\Gamma}_{k\ lm} + S_{m\ kl} \tag{3.6}$$

where $\hat{\Gamma}_{k\ lm}$ is given by (2.1). This connection is also the most general affine, metric (preserving) connection. We will be using this connection in all the future work, also because it is the only connection compatible with the equivalence principle.

We now give a geometrical interpretation of the contortion tensor. Parallel transferred basis vectors will differ from the original basis vectors in a point by

$$d\mathbf{e}_l = \nabla_m \mathbf{e}_l dx^m = \Gamma^k{}_{lm} \mathbf{e}_k dx^m \tag{3.7}$$

For an orthonormal anholonomic base, this difference will be

$$d\mathbf{e}_l = \left(\gamma_m{}^k{}_l + S_m{}^k{}_l\right)\mathbf{e}_k dx^m \tag{3.8}$$

Equation (3.8) expresses rotation of the parallel transferred orthonormal anholonomic basis with respect to the original orthonormal anholonomic basis. The first term expresses the (Ricci) rotation due to the Riemann structure (Riemannian metric) of the manifold, and the second term the rotation due to the torsional structure (cf. Hehl et al., 1976, p. 398).

Because it expresses rotation on parallel transfer even in a flat manifold, the contortion tensor is of great physical importance. As has been demonstrated in other papers (Gogala, 1980a, 1980b), the contortion tensor can describe noninertial motions, in particular, uniformly accelerated and uniformly rotating motions, as parallel transfer of the moving body's anholonomic basis (reference frame) along its world line; thus the contortion tensor is an excellent candidate for the geometrical interpretation of the electromagnetic field.

4. THE MATHEMATICAL ASPECTS OF THE PRINCIPLE OF EQUIVALENCE

We now briefly examine the equivalence principle for a manifold endowed with torsion. This principle was first discussed by von der Heyde (1975). We approach it here a little differently, and only from a merely geometrical standpoint. We take it in the following form: "The manifold must be locally Euclidean."

We do not specify, however, whether the local basis is coordinate or anholonomic. As has been pointed out by von der Heyde, the restriction of the equivalence principle to holonomic local bases introduces nonlocality, which contradicts the local character of the equivalence principle. The holonomity of the local basis implies that it corresponds to a global coordinate system.

We thus assume that: (i) the manifold is locally Euclidean, and (ii) the local basis is anholonomic, in general.

When we perform a transformation from the local anholonomic basis to a global coordinate basis, the components of the affine connection, which are zero in the local basis, because of (i), become

$$\Gamma^{K}{}_{LM} = h_k{}^{K}\left(h^{l}{}_{L} h^{m}{}_{M} \Gamma^{k}{}_{lm} + h^{k}{}_{L/M}\right) = h_k{}^{K} h^{k}{}_{L/M} \tag{4.1}$$

Since the transformation is anholonomic, because of assumption (ii), the connection is not symmetric in the coordinate basis. It is, however, still metric (preserving); the covariant derivative $\nabla_m g_{kl}$ is namely a tensor, and if it is zero locally, because of (i), it must be zero in the global coordinate basis too. Thus the only connection which is compatible with the equivalence principle, is a metric (preserving) connection with torsion (3.6), that is, Cartan's connection. Because the components of the affine connection are zero in the local basis, it is evident from (2.7) that the components of the torsion tensor are equal, up to the sign, to the components of the object of anholonomity in that basis. We must be careful not to confuse the two; their equality is due to the fact that all the components of the affine connection are zero in the local basis.

We also see from (4.1) that in the same way as the Riemannian metric arises from the transformation coefficients between the local Euclidean basis and the basis of the global coordinate system, via $g_{KL} = \eta_{kl} h^{k}{}_{K} h^{l}{}_{L}$, the torsion arises from the derivatives of these coefficients, when one transforms from the local into a coordinate basis. We also observe that it is impossible to determine from purely local information about the components of the affine connection in an anholonomic basis how much of the antisymmetric part (in the first two indices) of the affine connection is a Ricci rotation coefficient, thus belonging to the Riemannian structure of the manifold, and how much is a component of the contortion tensor. This is, in a way, similar to the impossibility of telling, from a purely local information about the metric tensor, whether globally the metric is Euclidean or Riemannian.

We now consider the reverse problem. It is well known that we can always find a special coordinate basis in which all the components of the Levi-Civita connection will be locally (in a point) equal to zero. One only needs to solve the transformation equations

$$\Gamma^{I'}{}_{J'N'} = h^{I'}{}_{K}\left(h_{J'}{}^{L} h_{N'}{}^{M} \Gamma^{K}{}_{LM} + h_{J'}{}^{K}{}_{/N'}\right) = 0 \tag{4.2}$$

for the unknowns $h_{J'}{}^{K}{}_{/N'}$. There are $n^2(n+1)/2$ equations, and n^3 unknowns, which are, however, restricted by $n^2(n-1)/2$ integrability conditions

$$h_{J'}{}^{K}{}_{/N'} - h_{N'}{}^{K}{}_{/J'} = 0 \tag{4.3}$$

reducing the number of unknowns to $n^2(n+1)/2$. Thus the number of equations is equal to the number of unknowns.

At first sight it seems that for the Cartan connection one cannot do the same thing, as there are now n^3 equations

$$\Gamma^{k}{}_{lm} = h^{k}{}_{K}\left(h_{l}{}^{L} h_{m}{}^{M} \Gamma^{K}{}_{LM} + h_{l}{}^{K}{}_{/m}\right) = 0 \tag{4.4}$$

This is true, however, only when one insists on a holonomic local basis. If one allows an anholonomic local basis, then there are no restrictions such as (4.3), and consequently there are also n^3 unknowns, so that all the components of the Cartan connection can be made to vanish locally, as was apparently first observed by von der Heyde (1975).

5. THE DISTORTION TENSOR

In a torsionless manifold, we have only one tensor quantity involving derivatives of the affine connection components. This is the curvature (tensor), defined as (in this paragraph, upper case letters denote vectors, and have nothing to do with coordinate basis)

$$\mathbf{Q}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \hat{\nabla}_{\mathbf{X}}\hat{\nabla}_{\mathbf{Y}}\mathbf{Z} - \hat{\nabla}_{\mathbf{Y}}\hat{\nabla}_{\mathbf{X}}\mathbf{Z} - \hat{\nabla}_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} \tag{5.1}$$

In a manifold with torsion, the situation is more complex because the affine connection can now be split into the Levi-Civita connection and the contortion. Besides the curvature, we now have the Riemann (tensor), defined as

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} \tag{5.2}$$

In addition, the difference of these two tensors is also a tensor:

$$\mathbf{P}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} - \mathbf{Q}(\mathbf{X},\mathbf{Y})\mathbf{Z} \tag{5.3}$$

On a torsionless manifold, Riemann and curvature tensors coincide. On a flat manifold, on the other hand, curvature is zero, although in general, $\mathbf{P}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}$ is nonzero. We now cannot talk about curvature, as the manifold is flat. The sense of parallelism is, however, greatly distorted on such a manifold. We therefore choose to call **P(X, Y)Z** the *distortion* (*tensor*). It is important to stress again that curvature is caused by the non-Euclidean metrical field on the manifold, whereas distortion is caused only by the presence of the torsional (or contortional) field on the manifold. It is true that the partial derivatives of the components of the metric tensor appear in (5.8), but that is only to guarantee the covariance of the expression.

The expression of Riemann and curvature tensors in components are well known, e.g.,

$$R^{w}{}_{z\,xy} = \Gamma^{w}{}_{zy/x} - \Gamma^{w}{}_{zx/y} + \Gamma^{w}{}_{mx}\Gamma^{m}{}_{zy} - \Gamma^{w}{}_{my}\Gamma^{m}{}_{zx} - \Gamma^{w}{}_{zm}c^{m}{}_{xy} \tag{5.4}$$

We now try to express the distortion tensor in component form, which we expect to be in terms of the contortion tensor components. From the definition (5.3), it follows that

$$\mathbf{P}(\mathbf{X},\mathbf{Y})\mathbf{Z} = (\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \hat{\nabla}_{\mathbf{X}}\hat{\nabla}_{\mathbf{Y}})\mathbf{Z} - (\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \hat{\nabla}_{\mathbf{Y}}\hat{\nabla}_{\mathbf{X}})\mathbf{Z} - (\nabla_{[\mathbf{X},\mathbf{Y}]} - \hat{\nabla}_{[\mathbf{X},\mathbf{Y}]})\mathbf{Z} \tag{5.5}$$

The first term can be expressed as

$$\begin{aligned}(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \hat{\nabla}_{\mathbf{X}}\hat{\nabla}_{\mathbf{Y}})\mathbf{Z} &= (\nabla_{\mathbf{X}} - \hat{\nabla}_{\mathbf{X}})(\nabla_{\mathbf{Y}} - \hat{\nabla}_{\mathbf{Y}})\mathbf{Z} + \hat{\nabla}_{\mathbf{X}}(\nabla_{\mathbf{Y}} - \hat{\nabla}_{\mathbf{Y}})\mathbf{Z} + (\nabla_{\mathbf{X}} - \hat{\nabla}_{\mathbf{X}})\hat{\nabla}_{\mathbf{Y}}\mathbf{Z} \\ &= S_{x}{}^{w}{}_{u}S_{y}{}^{w}{}_{z}\mathbf{e}_{w} + \left(S_{y}{}^{w}{}_{z/x} + S_{y}{}^{u}{}_{z}\hat{\Gamma}^{w}{}_{ux}\right)\mathbf{e}_{w} + S_{x}{}^{w}{}_{u}\hat{\Gamma}^{u}{}_{zy}\mathbf{e}_{w} \end{aligned} \tag{5.6}$$

when axioms (1.7) and expression (3.1) are used. The second term can be expressed in a similar way. The third term becomes

$$c^{u}{}_{xy}(\nabla_{\mathbf{u}} - \hat{\nabla}_{\mathbf{u}})\mathbf{Z} = c^{u}{}_{xy}S_{u}{}^{w}{}_{z}\mathbf{e}_{w} = \left(\hat{\Gamma}^{u}{}_{yx} - \hat{\Gamma}^{u}{}_{xy}\right)S_{u}{}^{w}{}_{z}\,\mathbf{e}_{w} \tag{5.7}$$

as Levi-Civita connection is torsionless.

When we insert the above expressions into (5.5), we get

$$\mathbf{P}_{wz\,xy} = \mathbf{W}\cdot\mathbf{P}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \left(S_{y\,wz;x} - S_{x\,wz;y}\right) + \left(Syw^{u}S_{x\,zu} - Syz^{u}S_{x\,wu}\right) \tag{5.8}$$

Here, ";" is the covariant derivative with the Levi-Civita connection.

Up to this point, no assumption yet has been made about the connection being metric, so the result (5.8) is quite general; it is valid as long as the connection $\hat{\nabla}$ is torsionless. If we now enforce the condition (3.4), that is, that the contortion tensor is antisymmetric in the last two indices, we find that $P_{wz\,xy}$, and consequently, $R_{wz\,xy}$ are antisymmetric in the first two indices besides being antisymmetric in the last two indices.

The latter antisymmetry follows from their definitions, whereas the former one is common to all Riemann tensors with a metric connection, as can be shown quite generally (Misner, 1969, p. 136; Misner et al., 1973, exercise 13.8).[4]

From the expression for the distortion tensor (5.8), we see, that in some special cases all its components may be identically zero, despite the manifold being endowed with nonzero torsion, which does not need to be constant either. This distinguishes the distortion tensor from the curvature tensor, whose components are all zero only if the metric is Euclidean. In many other ways the curvature and the distortion tensors are, however, analogous to each other, although they arise from different geometrical structures on the manifold. That has physical implications also (see Gogala 1980a, 1980b).

It must be stressed, however, that neither the components of the Riemann nor the distortion tensor are symmetric under the interchange of the first and second pair of indices unlike the components of the curvature tensor. Consequently, they have $[n(n-1)/2]^2$ components each. Neither do Ricci nor Bianchi identities in their usual form apply any more. Instead, they are generalized to read

$$R_{a[b\,cd]} = P_{a[b\,cd]} = T_{a[bc;d]} + T_{[b\,c}{}^{m}S_{d]am} \tag{5.9}$$

$$R_{ab[cd;e]} = P_{ab[cd;e]} = R^{n}{}_{a[cd}S_{e]bn} - R^{n}{}_{b[cds}S_{e]an} \tag{5.10}$$

where $[\cdots]$ denotes circular permutation of the indices.

As already mentioned, the sense of parallelism is greatly distorted in a manifold with torsion. We are intuitively used to extremals (geodesics) in a plane being also the straightest lines (autoparallels). This is no longer the case when a manifold is endowed with torsion. The straightest lines can now be spirals or some other curves, and appear "bent" to our intuition.

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[4]Although both references state that the proof is valid for the Levi-Civita connection only, one can see from the derivation that the proof is valid for any metric (preserving) connection.

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